MOMENTUM INTEGRAL METHOD FOR FORCED CONVECTION IN THERMAL NONEQUILIBRIUM POWER-LAW FLUID-SATURATED POROUS MEDIA

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Forced convection heat transfer for power-law fluid flow in porous media was studied analytically. The analytical solutions were obtained based on the Brinkman-extended Darcy model for fluid flow and the two-equation model for forced convection heat transfer. As a closed-form exact velocity profile is unobtainable for the general power-law index, an approximate velocity profile based on the parabolic model is proposed by subscribing to the momentum boundary layer integral method. Heat transfer analysis is based on the two-equation model by considering local thermal nonequilibrium between fluid and solid phases and constant heat flux boundary conditions. The velocity and temperature distributions obtained based on the parabolic model were verified to be reasonably accurate and improvement is justified compared to the linear model. The expression for the overall Nusselt number was derived based on the proposed parabolic model. The effects of the governing parameters of engineering importance such as Darcy number, power-law index, nondimensional interfacial heat transfer coefficient, and effective thermal conductivity ratio on the convective heat transfer characteristics of non-Newtonian fluids in porous media are analyzed and discussed.

Keywords Brinkman-extended darcy equation; Local thermal nonequilibrium; Momentum integral boundary layer; Power-law fluid

Introduction

Heat and fluid transport in porous media has been a subject of constant interest for several decades, mainly due to the wide range of heat transfer engineering applications, involving porous insulation, transpiration cooling, heat pipe wicking structures, oil recovery, packed-bed catalytic nuclear reactors, and even the flow of liquids in biological and physiological processes. More recently, the utilization of the porous media approach in the modeling of transport phenomena of microchannel heat sinks has received much interest as well due to its feasibility and accuracy. General information related to heat and fluid flow in porous media has been well documented, such as the studies by Kaviany (1995), Vafai (2005), and Nield and Bejan (2006). Most analytical studies in the literature deal primarily

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on the Newtonian fluid flow based on either Darcy’s law or modified Darcy’s law, which regards the boundary and inertia effects. However, more often than not, flow of complex fluids, primarily non-Newtonian fluids, is encountered as well in the aforementioned contemporary engineering applications.

Most initial studies of convection heat transfer involving non-Newtonian fluid flow are based on the Darcy equation for power-law fluid. Investigations were conducted on the effects of rheological parameters of the temperature-dependent non-Newtonian fluids on the heat and fluid flow in a porous medium (Wang and Tu, 1989; Pascal, 1990). Pascal (1992) solved the problem of the unsteady Darcian flow of non-Newtonian fluids of power-law behavior through a porous medium in a plane radial geometry with the similarity transformation method. Employing the same method, Nakayama and Shenoy (1992) and Shenoy (1994) solved the forced convective heat transfer problem using the Darcy model with and without the flow inertia term. Pearson and Tardy (2002) developed potential non-Newtonian and complex fluids models to represent the flow in a porous medium by using Darcy’s law for power-law fluids.

On the other hand, several studies were attempted to investigate the heat and fluid flow characteristics of a non-Darcian non-Newtonian fluid in porous media. By applying a volume-averaging approach in deriving a non-Darcian model, Liu and Masliyah (1999) predicted the velocity profile and pressure drop of non-Newtonian fluid flow through orifice plates utilizing a shear factor model. Tsakiroglou (2002) employed the approximate Rabinowitch-Mooney equation to numerically simulate the flow of non-Darcian inelastic non-Newtonian fluids over the creeping flow regime in porous media. Non-Darcian forced convection problems of power-law fluids in porous media were solved primarily numerically by Chen and Hadim (1998a, 1998b, 1999a) due to the prevailing strong nonlinear effects. Al-Nimr and Aldoss (2004) solved the one-dimensional unsteady Darcy-Brinkman-Forchheimer equation numerically to investigate the effect of the macroscopic local inertial term on the non-Newtonian fluid flow in horizontal parallel-plate channels filled with a porous medium. Taking into account the boundary frictional and flow inertia effects, Nakayama and Shenoy (1993) conducted a more comprehensive study on convective heat transfer of a non-Newtonian fluid in porous media, in which the Brinkman-Forchheimer-extended Darcy equation was solved numerically and the possibility of obtaining a similarity solution for the special case of very low Darcy number was suggested. Ellahi and Afzal (2009) derived a comprehensive mathematical model for the flow of temperature-dependent non-Newtonian fluids, such as third-grade fluids, in a porous medium, using a modified Darcy’s law. The problems were solved analytically to obtain the series solutions of velocity and temperature by using the homotopy analysis method. Most recently, Abel et al. (2011) investigated the heat transfer characteristics of non-Newtonian fluid flow over a stretching sheet through a porous medium under the effect of external magnetic field.

The momentum integral boundary layer method has been employed extensively to obtain approximate solutions for boundary layer problems when an exact solution does not exist. The various integral methods essentially differ in the prescribed profile family and the use of different integral relations (Schlichting and Gersten, 2000). Integral solutions were obtained for forced convection of Newtonian fluid flow over a semi-infinite flat plate embedded in a porous medium (Kaviany, 1987). Nakayama et al. (1988) solved the Brinkman-Darcy forced convection problem in a channel filled with porous media saturated with Newtonian fluids using the
integral method. By applying the Brinkman-Forchheimer-extended Darcy model, Li et al. (2009) obtained the integral solution of a forced laminar boundary layer over a plate embedded in a porous medium. For convective heat transfer of a non-Newtonian fluid in porous media, Nakayama and Shenoy (1993) employed the momentum integral boundary layer method to approximate the velocity profile using a linear model. By means of the approximate method in obtaining the velocity profile proposed by Nakayama and Shenoy (1993), Nield and Kuznetsov (2005) solved the constant wall heat flux problem for the case of thermally developing forced convection of non-Newtonian fluid flow in a porous medium. Employing the integral method with a parabolic velocity profile, Chen and Hadim (1999b) obtained analytical solutions for heat and fluid flow in a power-law fluid–saturated porous parallel plate channel. The flow was modeled by the Brinkman-Forchheimer-extended Darcy equation while the convection heat transfer in a porous medium was analyzed using the one-equation model, assuming the fluid phase is in local thermal equilibrium with the solid phase.

All the aforementioned studies used the one-equation model for the energy equation in analyzing convection heat transfer through porous media under local thermal equilibrium. This model has been employed in various heat transfer investigations of Newtonian fluids in porous media, such as those by Nield et al. (1996), Hunt and Tien (1998), Haji-Sheikh et al. (2004, 2005), and Hung and Tso (2008, 2009), to name a few. For cases where the fluid and solid phases are not in local thermal equilibrium, the two-equation model, which characterizes the evolution of the temperature of each phase in a porous medium, is employed. Considerable attention has been given to problems of local thermal nonequilibrium in porous media in the light of the extensive contemporary engineering applications, such as microchannel heat sinks, heat pipes, nuclear reactors, and others. Vafai and coworkers (Amiri et al., 1995; Lee and Vafai, 1999; Alazmi and Vafai, 2002) analyzed boundary conditions for model equations that govern local thermal nonequilibrium phenomena of convective heat transfer in a porous medium under constant wall heat flux. The previous studies employing the two-equation model (Amiri and Vafai, 1994; Amiri et al., 1995; Lee and Vafai, 1999; Alazmi and Vafai, 2002; Kim et al., 2000; Marafie and Vafai, 2001; Nield et al., 2002) essentially dealt with analyses of forced convection heat transfer involving Newtonian fluid flow in porous media; those involving non-Newtonian fluids are very scarce in the existing literature. Only recently, Khashan and Al-Nimr (2005) applied the Brinkman-Forchheimer-extended Darcy equation with a two-equation model to study the forced convection heat transfer in a porous medium filled with a power-law fluid, and the governing equations were solved subscribing to the numerical method. As pointed out by Nakayama and Shenoy (1993), forced convection for non-Newtonian fluid flow in a porous medium involves a number of dimensionless parameters such that numerical investigation of the problem demands lengthy numerical integrations before analyzing the heat transfer characteristics. In view of this, it is more appealing to solve the problem analytically for the sake of simplicity in performing a parametric study of the complex combined effects of these parameters.

The present study emphasizes analytical solutions for the velocity and temperature distributions as well as the Nusselt number for forced convection of power-law fluids in porous media under local thermal nonequilibrium. Although numerous studies have been carried out on forced convection of power-law fluids in porous media, analytical results for the case of thermal nonequilibrium have not been reported in the literature. To analytically solve the Brinkman-extended
Darcy equation modified for a power-law fluid, an improved approximate solution method by subscribing to the momentum integral boundary layer technique assuming a parabolic velocity profile for the power-law fluid is newly proposed. An explicit expression of the boundary layer thickness of the momentum equation is obtained. For the energy balance analysis, under local thermal nonequilibrium and constant heat flux boundary conditions, the two-equation model is employed to model the forced convective heat transfer in porous media. With variations of the dimensionless parameters, the behavior of the velocity and temperature distributions as well as the Nusselt number are analyzed and discussed.

Mathematical Formulation

Momentum Equation

The problem under investigation in the present study is steady laminar forced convective flow of a non-Newtonian fluid through a fixed infinitely long parallel-plate channel distanced \( d \) apart and filled with a porous medium, which is subjected to constant heat fluxes, \( q_w \), at both channel walls. The non-Newtonian fluid is modeled to be a power-law fluid using the Ostwald-de Waele model with constant fluid properties under hydrodynamically and thermally fully developed conditions. The Brinkman-extended Darcy equation for the flow of a power-law fluid is expressed as (Nakayama and Shenoy, 1993)

\[
\frac{\mu^*}{\varepsilon^n} \frac{d}{dy} \left[ \left( \frac{d\langle u \rangle}{dy} \right)^{n-1} \left( \frac{d\langle u \rangle}{dy} \right) \right] - \frac{\mu^*}{K^*} \langle u \rangle^n - \frac{d\langle p \rangle_f}{dx} = 0
\]

(1)

where \( u \) is the fluid velocity, \( \varepsilon \) is the porosity, \( \mu^* \) denotes the fluid consistency of the inelastic non-Newtonian power-law fluid, \( K^* \) is the modified permeability of the porous medium for flow of a power-law fluid, and \( n \) is the power-law index where the fluid is shear thinning or pseudoplastic for \( 0 < n < 1 \), Newtonian for \( n = 1 \), and shear thickening or dilatant for \( n > 1 \). The symbols \( \langle \cdot \rangle \) and \( \langle \cdot \rangle_f \) represent the volume-averaged value over the entire porous structure and fluid region, respectively, and the applied pressure gradient \( d\langle p \rangle_f/dx \) is taken to be a constant. The velocity in Equation (1) is volume-averaged over the entire porous structure, denoted as the Darcian velocity or superficial velocity. To be consistent with the energy equations, the Darcian velocity is converted to the velocity which is volume-averaged over the fluid region using the Dupuit-Forchheimer relationship as (Nield and Bejan, 2006)

\[
\langle u \rangle = \varepsilon \langle u \rangle_f
\]

(2)

After substituting Equation (2) into Equation (1), the Brinkman-extended Darcy equation modified for a power-law fluid is transformed to the form of

\[
\frac{\mu^*}{\varepsilon^n} \frac{d}{dy} \left[ \left( \frac{d\langle u \rangle_f}{dy} \right)^{n-1} \left( \frac{d\langle u \rangle_f}{dy} \right) \right] - \frac{\mu^*}{K^*} \varepsilon^n \langle u \rangle_f^n - \frac{d\langle p \rangle_f}{dx} = 0
\]

(3)
and the nondimensional form of Equation (3) is given by

\[ U|U|^{n-1} = Da^{(n+1)/2} \frac{d}{dY} \left[ \left| \frac{dU}{dY} \right|^{n-1} \left( \frac{dU}{dY} \right) \right] + P \]  

(4)

where

\[ U = \frac{<u>_f}{u_m}, \quad Y = \frac{y}{d}, \quad P = \frac{K^*}{\mu u_m e^*} \left( -\frac{d\langle p \rangle_f}{dx} \right), \quad Da = \frac{(K^*/\epsilon)^{2/(n+1)}}{d^2}, \]  

(5)

with \( u_m \) the mean velocity in the fluid region and \( Da \) the modified Darcy number for power-law fluids. A closed-form analytical solution for Equation (4) is not obtainable for the case of general \( n \) due to its inherent nonlinear nature. By applying the momentum integral boundary layer technique, Nakayama and Shenoy (1993) assumed a linear profile, while Chen and Hadim (1999b) used a parabolic profile to obtain an approximate solution to the velocity profile of the Brinkman-Forchheimer-extended Darcy law modified for power-law fluids. As commonly known, the more closely the assumed shape approximates the actual profile of the exact solution, the more accurate the results generated. To this end, we propose an improved approximate solution method by assuming a parabolic velocity profile for the power-law fluid in the momentum integral boundary layer technique. Equation (4) is integrated over the half-channel to give

\[ -\left| \frac{dU}{dY} \right|^{n-1} \left( \frac{dU}{dY} \right)^n \bigg|_{Y=0} = \frac{1}{Da^{(n+1)/2}} \int_0^{1/2} \left( U|U|^{n-1} - P \right) dY \]  

(6)

In deriving Equation (6), the symmetry condition \( dU/dY = 0 \) at \( Y = 1/2 \) is utilized. For a highly porous medium, the viscous effect is confined to a thin layer from the channel plate, and outside this boundary layer, the velocity is nearly invariable across the channel, denoted as the centerline velocity (Nakayama and Shenoy, 1993). In accord with Equation (4), the dimensionless form of the centerline velocity is given explicitly by

\[ U_c = P^{1/n} \]  

(7)

In line with this, the appropriate boundary conditions for the velocity profile across the boundary layers are

\[ U(0) = U(1) = 0, \quad U(\delta) = U(1 - \delta) = U_c, \quad \frac{dU}{dY} \bigg|_{Y=\delta} = \frac{dU}{dY} \bigg|_{Y=1-\delta} = 0 \]  

(8)

where \( \delta \) is the dimensionless form of the boundary layer thickness. Application of the boundary conditions in Equation (8) gives the general velocity profile across the channel, which is characterized by a quadratic function in piecewise form as

\[ U = \begin{cases} 
U_c \left[ 2\left( \frac{Y}{\delta} \right) - \left( \frac{Y}{\delta} \right)^2 \right], & 0 \leq Y \leq \delta, \\
U_c, & \delta \leq Y \leq 1 - \delta, \\
U_c \left[ -2\left( \frac{Y-1}{\delta} \right) - \left( \frac{Y-1}{\delta} \right)^2 \right], & 1 - \delta \leq Y \leq 1.
\end{cases} \]  

(9)
By substituting Equation (9) into Equation (6) and resolving the dimensionless boundary layer thickness $\delta$, its explicit expression in terms of gamma function $\Gamma$ can be evaluated as

$$\delta = \frac{2\pi \Gamma(n+1)/\Gamma(n+3/2)^{1/(n+1)}}{[1 - \sqrt{\pi} \Gamma(n+1)/2 \cdot \Gamma(n+3/2)]^{1/(n+1)}}$$

(10)

The definition of the mean velocity, $u_m = \int_0^d u \, dy/d$, leads to the relation

$$\int_0^1 U \, dY = 1$$

(11)

Integrating Equation (9) and utilizing the relation in Equation (11), the dimensionless centerline velocity is given by

$$U_c = \frac{3}{3 - 2\delta}$$

(12)

In the present study, we obtain explicit expressions for the boundary layer thickness and centerline velocity, compared to those by Chen and Hadim (1999b), where the boundary layer thickness was expressed in the form of an infinite series and the coupled equations of boundary layer thickness and centerline velocity were solved iteratively using the Gauss-Seidel method. Following Equations (7) and (12), the dimensionless pressure gradient for the parabolic model can be expressed as

$$P = \left( \frac{3}{3 - 2\delta} \right)^n$$

(13)

Consequently, an approximate solution by appealing to the momentum integral method with a parabolic shape for the velocity profile of a power-law fluid is obtained. This integral treatment to obtain the approximate velocity profile is applicable to a highly porous medium with small Darcy number, which is usually of practical engineering interest.

**Energy Equations**

Under local thermal nonequilibrium and constant heat flux boundary conditions, the steady-state volume-averaged energy equations for the solid and fluid phases are, respectively, as given by Tien and Kuo (1987):

$$(1 - \varepsilon)k_s\frac{\partial^2 (T)_s}{\partial y^2} = h_{sf}a((T)_s - (T)_f)$$

(14)

$$\varepsilon \rho_f c_f \frac{\partial (T)_f}{\partial x} = h_{sf}a((T)_s - (T)_f) + \varepsilon \rho_f c_f \frac{\partial^2 (T)_f}{\partial y^2}$$

(15)

where $a$, $\rho_f$, and $c_f$ are specific surface area, density of fluid, and heat capacity, respectively, and $k_s$ and $k_f$ are thermal conductivity of solid and thermal conductivity
of fluid, respectively. The interfacial heat transfer coefficient $h_{sf}$ is the proportionality constant between the interfacial heat flux and the solid-fluid temperature difference, and $\langle \cdot \rangle_s$ refers to the volume-averaged value over the solid region. The appropriate boundary conditions for Equations (14) and (15) are defined based on the temperature continuity at the wall as (see Amiri et al., 1995; Lee and Vafai, 1999; Alazmi and Vafai, 2002)

$$\langle T \rangle_s = \langle T \rangle_f = T_w \quad \text{at} \quad y = 0, d$$

where $T_w$ is the wall temperature. Nield and Kusnetsov (1999) pointed out that these boundary conditions arise due to the idealization of a perfectly conducting solid substrate of finite thickness for the channel wall. This assumption allows the temperatures at the interface between the porous medium and the solid wall to be uniform, regardless of which phase comes into contact with the wall (Lee and Vafai, 1999). In addition to Equation (16), the thermal boundary condition at the wall is defined by the energy balance at the interface of the wall and the porous medium. Alazmi and Vafai (2002) discussed various models for uniform wall heat flux. In the present study, we employ the model of uniform flux over the two phases, which is the most commonly adopted and considered as the best model among others, as pointed out by Nield and Bejan (2006). The heat flux applied at the wall is apportioned between the solid and fluid phases based on their effective thermal conductivity and temperature gradient as (see Amiri et al., 1995; Lee and Vafai, 1999)

$$q_w = -\left(\varepsilon k_f \frac{\partial \langle T \rangle_f}{\partial y}\bigg|_{y=0} + (1 - \varepsilon)k_s \frac{\partial \langle T \rangle_s}{\partial y}\bigg|_{y=0}\right)$$

$$= \varepsilon k_f \frac{\partial \langle T \rangle_f}{\partial y}\bigg|_{y=d} + (1 - \varepsilon)k_s \frac{\partial \langle T \rangle_s}{\partial y}\bigg|_{y=d}$$

which yields the surface energy balance at the walls. By summing up Equations (14) and (15), then integrating it over the channel cross section and applying the boundary conditions in Equation (17), the energy balance requires that

$$\frac{\partial \langle T \rangle_f}{\partial x} = \frac{2q_w}{\varepsilon \rho_f c_f u_m d} = \alpha$$

Equation (18) demonstrates that the axial fluid temperature gradient is reduced to a constant $\alpha$. Under a fully developed thermal condition with constant heat flux boundary condition, the longitudinal conduction term is absent in the energy equations since its contribution to the net energy transfer is negligible. In this case, the temperature gradient along the axial direction is independent of the transverse direction and can be written as

$$\frac{\partial \langle T \rangle_f}{\partial x} = \frac{\partial \langle T \rangle_s}{\partial x} = \frac{dT_w}{dx} = \alpha$$

From Equation (19), it can be deduced that the fluid and solid temperature distributions take the form of $\alpha x + f(y)$, with the function of $f(y)$ to be determined by solving Equations (14) and (15). It follows that the temperature difference between the wall and the solid or fluid phase must be independent of $x$ in the axial direction, regardless of the variation in the parameters. Therefore, for further heat transfer analysis, it is
more appealing to solve for the temperature distributions in the transverse direction of \( y \). To proceed, the dimensionless temperatures for each phase are introduced as

\[
\theta_s = \frac{\langle T \rangle_s - T_w}{2q_w d/(1 - \epsilon)k_s}, \quad \theta_f = \frac{\langle T \rangle_f - T_w}{2q_w d/(1 - \epsilon)k_s}
\]

(20)

where \( q_w \) is the uniform heat flux applied at the impermeable walls. It is noted that \( \theta_s \) and \( \theta_f \) are functions of \( y \) only due to the fact that the differences \( \langle T \rangle_s - T_w \) and \( \langle T \rangle_f - T_w \) are independent of \( x \) along the axial direction. By utilizing Equations (18) and (19), Equations (14)–(16) can be nondimensionalized as

\[
d^2\theta_s/dY^2 = H(\theta_s - \theta_f)
\]

(21)

\[
U = H(\theta_s - \theta_f) + \gamma d^2\theta_f/dY^2
\]

(22)

\[
\theta_s = \theta_f = 0 \quad \text{at} \quad Y = 0, 1
\]

(23)

where

\[
\gamma = \frac{\epsilon k_f}{(1 - \epsilon)k_s}, \quad H = \frac{h_s a d^2}{(1 - \epsilon)k_s}
\]

(24)

In Equation (24), \( \gamma \) is the effective thermal conductivity ratio, and the nondimensional interfacial heat transfer coefficient \( H \) represents the ratio of the internal convection heat transfer between the solid and fluid to the conduction resistance of the solid (Marafiie and Vafai, 2001). With the velocity distribution for a power-law fluid given by Equation (9), the energy equations (21) and (22) are solved by imposing the requirement of continuity of temperature and heat flux at \( Y = \delta \) and \( Y = 1 - \delta \) and then applying the boundary conditions in Equation (23) to yield the closed-form dimensionless temperature distributions for fluid and solid as follows:

\[
\theta_s = \left\{ \begin{array}{ll}
\frac{U_c}{\gamma + 1} \left[ -\frac{1}{12\delta^2} Y^4 + \frac{1}{3\delta} Y^3 + \frac{1}{E\delta^2} Y^2 \\
+ \left( \frac{2\delta - 3}{6} - \frac{2}{E\delta^2} \right) Y + C_1 e^{\delta Y} - C_2 e^{-\delta Y} + \frac{2\gamma}{E^2\delta^2} \right], & 0 \leq Y \leq \delta, \\
\frac{U_c}{\gamma + 1} \left[ \frac{1}{2} Y^2 - \frac{1}{2} Y + C_3 e^{\delta Y} - C_4 e^{-\delta Y} + \left( \delta^2 - \frac{1}{E\delta^2} \right) \right], & \delta \leq Y \leq 1 - \delta,
\end{array} \right.
\]

\[
\theta_f = \left\{ \begin{array}{ll}
\frac{U_c}{\gamma + 1} \left[ -\frac{1}{12\delta^2} Y^4 + \frac{1}{3\delta} Y^3 + \frac{1}{E\delta^2} Y^2 \\
+ \left( \frac{2\delta - 2}{E\delta^2} - \frac{2\delta^3 - 3\delta^2 + 6\delta - 2}{6\delta^2} \right) Y \\
+ C_5 e^{\delta Y} - C_6 e^{-\delta Y} + \frac{4\delta^3 - 6\delta^2 + 4\delta - 1}{12\delta^2} \right], & 0 \leq Y \leq \delta, \\
\frac{2\gamma - E(2\delta - 1)}{E^2\delta^2}, & 1 - \delta \leq Y \leq 1,
\end{array} \right.
\]

(25)
\[ \theta_s = \begin{cases} 
\frac{1}{E \delta^2} Y^2 + \frac{2}{E \delta} Y - C_1 e^{\lambda Y} + C_2 e^{-\lambda Y} - \frac{2 \gamma}{E^2 \delta^2} & 0 \leq Y \leq \delta, \\
\frac{1}{\gamma + 1} \left[ - \frac{1}{12 \delta^2} Y^4 + \frac{1}{3 \delta} Y^3 + \frac{1}{E \delta^2} Y^2 \right] + \left( \frac{2 \delta - 3}{6} - \frac{2}{E \delta} \right) Y + C_1 e^{\lambda Y} - C_2 e^{-\lambda Y} + \frac{2 \gamma}{E^2 \delta^2} \right] \} 
\end{cases} \]

\[ U_c \left\{ - C_3 e^{\lambda Y} + C_4 e^{-\lambda Y} + \frac{1}{E} + \frac{1}{\gamma + 1} \left[ \frac{1}{2} Y^2 - \frac{1}{2} Y \right] \right\}, \]

\[ \delta \leq Y \leq 1 - \delta, \]

\[ U_c \left\{ - C_3 e^{\lambda Y} - C_4 e^{-\lambda Y} + \left( \frac{\delta^2}{12} - \frac{1}{E} \right) \right\}, \]

\[ 1 - \delta \leq Y \leq 1, \]

\[ \frac{4 \delta^3 - 6 \delta^2 + 4 \delta - 1}{12 \delta^2} + \frac{2 \gamma - E(2 \delta - 1)}{E^2 \delta^2} \right] \} \}, \]

and the coefficients appearing in the above expressions are given by

\[ C_1 = \frac{\gamma}{D} (2 - e^{\lambda \delta} - e^{-\lambda \delta} + e^{\lambda(1-\delta)} + e^{-\lambda(1-\delta)} - 2e^{-\lambda}) \]  

\[ C_2 = \frac{\gamma}{D} (2 - e^{\lambda \delta} - e^{-\lambda \delta} + e^{\lambda(1-\delta)} + e^{-\lambda(1-\delta)} - 2e^{\lambda}) \]  

\[ C_3 = \frac{\gamma}{D} (2 - e^{\lambda \delta} - e^{-\lambda \delta} + e^{\lambda(1+\delta)} + e^{-\lambda(1-\delta)} - 2e^{-\lambda}) \]  

\[ C_4 = \frac{\gamma}{D} (2 - e^{\lambda \delta} - e^{-\lambda \delta} + e^{\lambda(1+\delta)} + e^{-\lambda(1-\delta)} - 2e^{\lambda}) \]  

\[ C_5 = \frac{\gamma}{D} (2 - e^{-\lambda(2-\delta)} - e^{-\lambda \delta} + e^{\lambda(1+\delta)} + e^{-\lambda(1-\delta)} - 2e^{-\lambda}) \]  

\[ C_6 = \frac{\gamma}{D} (2 - e^{\lambda(2-\delta)} - e^{\lambda \delta} + e^{\lambda(1+\delta)} + e^{\lambda(1-\delta)} - 2e^{\lambda}) \]  

\[ D = E^2 \delta^2 (e^{-\lambda} - e^{\lambda}) \]  

\[ \lambda = \sqrt{\frac{E}{\gamma}} \]
\[ E = H(\gamma + 1) \]  

Subsequently, the overall heat transfer coefficient for the fluid phase is obtained as

\[ \bar{h} = \frac{2q_w}{T_w(x) - T_{fb}(x)} = -\frac{(1 - e)k_s}{d\theta_{fb}(x)} \]  

where \( \theta_{fb}(x) \) is the dimensionless bulk mean fluid temperature averaged over the cross section of the channel, given by

\[ \theta_{fb}(x) = \frac{T_{fb}(x) - T_w(x)}{2q_w d/(1 - e)k_s} = \frac{\int_0^1 U \theta_t \, dY}{\int_0^1 U \, dY} \]

and the overall Nusselt number can be derived as

\[ \text{Nu} = \frac{\bar{h}d}{\kappa_f} = -\frac{1}{\gamma \theta_{fb}} \]

\[ = 1/\gamma B_6 \{ C_1 (B_1 - B_2 e^{\beta}) + C_2 (B_3 + B_2 e^{-\beta}) \]

\[ - B_4 [C_3 (e^{\beta} - e^{\lambda (1-\beta)}) + C_4 (-e^{-\beta} + e^{-\lambda (1-\beta)})] \]

\[ + C_5 (B_3 e^{\lambda} + B_2 e^{\lambda (1-\beta)}) + C_6 (B_1 e^{-\lambda} - B_2 e^{-\lambda (1-\beta)}) + B_5 \}, \]

where

\[ B_1 = 2\gamma E(\sqrt{\gamma} + \delta \sqrt{E}) \]

\[ B_2 = \sqrt{\gamma} E(2\gamma - \delta^2 E) \]

\[ B_3 = 2\gamma E(\delta \sqrt{E} - \sqrt{\gamma}) \]

\[ B_4 = \delta^2 E^2 \sqrt{\gamma} \]

\[ B_5 = E^{5/2} \left( \frac{31}{315} \delta^5 - \frac{1}{6} \delta^4 + \frac{1}{12} \delta^2 \right) - E^{3/2} \left( \frac{14}{15} \delta^3 - \delta^2 \right) - E^{1/2} \left( \frac{8}{3} \gamma \delta \right) \]

\[ B_6 = \frac{U_t^2}{E^{5/2} \delta^2 (1 + \gamma)} \]

**Results and Discussion**

**Velocity Distributions**

In the present study, an approximate velocity profile of parabolic model with parabolic shape inside the boundary layer is proposed to model the non-Newtonian fluid flow in a channel filled with a porous medium. To prove the validity and improvement of this second-order model, a comparison among the parabolic model, the linear model, and the exact solution model for the case of Newtonian fluid \( (n = 1) \)
is performed. The linear model for power-law fluid proposed by Nakayama and Shenoy (1993) and the exact solution for the Newtonian fluid obtained by Kim et.al. (2000) are adopted in accordance with the fluid flow and thermal boundary conditions of the present study. For the sake of brevity, the solutions for the velocity and temperature profiles for both approaches are omitted here. The velocity profiles for these three models are plotted for the case of $n = 1$ with different Darcy numbers in Figure 1. It is observed that the velocity profile of the parabolic model traces closer to the exact solution than that of the linear model. This verifies that the parabolic model is a better approximation than the linear model. However, in the vicinity of the boundary layer, significant differences are still observed between the parabolic model and the exact solution and the differences are observed to increase with the Darcy number. Nonetheless, the maximum discrepancies are within 10%, which can be considered reasonably small in most engineering applications. It is also worth noting that the parabolic model is more realistic in nature as the velocity gradient does not change abruptly at the boundary layer thickness as that of the linear model does.

Figure 2(a) plots the boundary layer thickness (of the parabolic model) as a function of the Darcy number, with the power-law index $n$ being a parameter. The effect of varying the Darcy number can be observed by comparing Figures 1(a) and 1(b), where reducing the Darcy number results in a more uniform velocity profile due to the fact that the region sensitive to the boundary effect decreases. As illustrated in Figure 2(a), when the Darcy number decreases, the action of the viscosity is confined in a smaller region from the impermeable wall. Figure 2(b) shows the comparison between the numerical solutions of the velocity profile presented by Nakayama and Shenoy (1993) and the parabolic model for non-Newtonian fluids when $Da' = 0.01$. In this context, $Da'$ refers to the Darcy number defined by Nakayama and Shenoy (1993), which is equivalent to four times the Darcy number $Da$ defined in this article. The trend of the parabolic model agrees qualitatively and quantitatively with the linear model for both pseudoplastic and dilatant fluids. The parabolic model is also found to perform better for higher $n$ values as the distribution of higher $n$ traces closer to its counterpart of the numerical solution. Referring to Figure 2(a), we can observe that the boundary layer thickness decreases with $n$, hence it is shown in Figure 2(b) that the velocity profile becomes

![Figure 1. Comparison between dimensionless velocity distributions for exact solution, parabolic, and linear models for $n = 1$ when (a) $Da = 0.01$ and (b) $Da = 0.001$.](image-url)
more uniform and flatter as the power-law index \( n \) increases. This is an interesting observation because it is commonly known that the converse is true for non-Newtonian fluid flow in clear-fluid channel (Nakayama and Shenoy, 1993).

**Temperature Distributions**

The comparison among the dimensionless temperature profiles of the parabolic model, linear model (Nakayama and Shenoy, 1993), and exact solution (Kim et al., 2000) for \( n = 1 \) at \( Da = 0.01 \), \( H = 20 \), and \( \gamma = 0.5 \) is shown in Figure 3. It is observed that the parabolic model traces very closely to the exact solution compared to the linear model. The improvement is more prominent in the solid temperature where the parabolic model and the exact solution coincide. Such improvement in the

![Figure 3. Comparison among dimensionless temperature profiles for solid and fluid for exact solution, linear, and parabolic models for \( n = 1 \) when \( Da = 0.01 \), \( H = 20 \), and \( \gamma = 0.5 \).](image)
temperature distributions is directly attributable to the more accurate representation of the velocity profile by the parabolic model.

To investigate the effect of varying Darcy number for non-Newtonian fluids, the dimensionless temperature profiles of $\gamma = 0.05$ and $H = 20$ with Darcy number being a variable parameter are plotted in Figure 4. It is observed that the temperature difference between fluid and solid decreases as the Darcy number is reduced from 0.01 to 0.0001. This effect is not only observed in the pseudoplastic and dilatant fluids, but also in Newtonian fluid (Kim et al., 2000). Decreasing the Darcy number means that the pores are getting finer as there are more solid constituents present in the porous medium. This in turn results in an increase in specific surface area available for the heat transfer between the solid and fluid phases, besides increasing the interfacial heat transfer coefficient, where both of these are accountable for reducing the temperature difference between phases. Scrutiny of Figure 4 reveals that the fluid temperature appears to be flatter as the Darcy number is reduced for both types of fluid, induced by the direct manifestation of the effect of reducing Darcy number in the velocity profile, where it becomes flatter, as explained in the previous section. Comparing Figures 4(a) and 4(b), the fluid and solid temperature profiles for dilatant fluid are flatter and broader than those of pseudoplastic fluid, regardless of the Darcy number. Similarly, this phenomenon is due to the flattening effect on the velocity profile when $n$ increases, as depicted in Figure 2(b). Therefore, both the Darcy number and the power-law index influence the temperature distributions and hence the heat transfer characteristics of non-Newtonian flowing in a porous medium, as we shall discuss shortly.

Figure 5 plots the temperature profiles with $H = 20$, $Da = 0.01$, and various $\gamma$ values for $n = 0.5$ and $n = 2$. The effective thermal conductivity ratio represents the ratio of heat conductance between fluid and solid phases. Therefore, an increase in $\gamma$ refers to relative increase in the heat conducted through the fluid compared to that through the solid phase. It is observed that an increase in $\gamma$ intensifies the temperature of both solid and fluid phases. In addition, the temperature difference between the two phases is reduced when $\gamma$ is increased, due to the fact that the increasing amount of heat supplied from both ends of the channel is transferred directly to the fluid by conduction rather than going through the solid phase and then back to the fluid phase. Subsequently, the amount of heat being transferred

![Figure 4](image.png)

**Figure 4.** Dimensionless temperature profiles for $H = 20$ and $\gamma = 0.05$ and selected Darcy number values for (a) $n = 0.5$ and (b) $n = 2$. 
between the phases is reduced, which in turn results in a decrease in the temperature difference between the phases, as the Darcy number and nondimensional interfacial heat transfer coefficient remain constant.

For a given Darcy number (Da = 0.01) and effective thermal conductivity ratio ($\gamma = 0.05$), the dimensionless solid and fluid temperature profiles are presented in Figure 6 for $n = 0.5$ and $n = 2$, to examine the effect of varying nondimensional interfacial heat transfer coefficient $H$. For both dilatant and pseudoplastic fluids, the dimensionless temperature profiles of the solid and fluid phases appear to be closer to each other as $H$ is increased from 20 to 200. This results in a reduction of the temperature difference between the phases, while the dimensionless solid temperature hardly changes for both the dilatant and pseudoplastic fluids. To elucidate the effects of $H$, the effective thermal conductivity of solid, $(1 - \varepsilon)k_s$, specific surface area $a$, and the channel height $d$ is arbitrarily fixed. By doing so, we can observe that increasing $H$ actually increases the interfacial heat transfer coefficient, implying that a smaller temperature difference between the phases is required to maintain the heat transfer rate. Comparing Figures 5 and 6, it is found that increasing $n$ produces a flatter temperature profile. Again, the reason behind this trend is none other than the flattening effect of the velocity profile being reflected in the temperature distributions when $n$ increases.

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**Figure 5.** Dimensionless temperature profiles for Da = 0.01 and $H = 20$ and selected $\gamma$ values for (a) $n = 0.5$ and (b) $n = 2$.

**Figure 6.** Dimensionless temperature profiles for $\gamma = 0.05$ and Da = 0.01 and selected $H$ values for (a) $n = 0.5$ and (b) $n = 2$. 
Heat Transfer Characteristics

The overall Nusselt number is an important heat transfer characteristic that indicates the heat transfer performance between the fluid flow and the channel walls. From Equation (38), it is noticed that the overall Nusselt number is a function of Da, \( H \), \( \gamma \) and \( n \). Figure 7 plots the overall Nusselt number as a function of Darcy number with \( H = 200 \), \( \gamma \) being a parameter, for \( n = 0.5 \) and \( n = 2 \). The overall Nusselt number increases as the Darcy number decreases. This phenomenon can be understood better by referring to Figure 4, where the magnitude of the dimensionless fluid temperature becomes smaller (smaller wall-fluid temperature difference) when Da is decreased, leading to a decrease in the dimensionless bulk mean temperature \( \theta^{*}_{b} \), and subsequently from Equation (38), the overall Nusselt number is found to be increased.

Figure 8 depicts the overall Nusselt number as a function of nondimensional interfacial heat transfer coefficient for \( Da = 0.01 \) and various \( \gamma \) values, when \( n = 0.5 \) and \( n = 2 \). It is observed that \( \bar{Nu} \) increases drastically from small \( H \) and approaches the asymptotic value when \( H \) exceeds \( 10^2 \). This observation is attributable to the fact that increasing \( H \) induces a decrease in the temperature difference between the fluid and the solid phases, and in the wall-fluid temperature difference as well, as shown in Figure 6. At higher \( H \), the dimensionless fluid temperature approaches closer to the dimensionless solid temperature, which is insensitive to the variation of \( H \). The overall Nusselt number saturates at a high \( H \) value due to the fact that it is impossible for the fluid temperature to exceed the solid temperature, which serves as the limiting factor, leading to an asymptotic value of the wall-fluid temperature difference and the Nusselt number as well. One common observation that can be found in Figures 7 and 8 is that \( \bar{Nu} \) increases with decreasing \( \gamma \), as can be readily understood from the expression of \( \bar{Nu} \) in Equation (38) where \( \bar{Nu} \) is inversely proportional to \( \gamma \). The dependence of the overall Nusselt number on the power-law index is depicted in Figure 9 when \( Da = 0.01 \), \( H = 20 \), and \( \gamma = 0.05 \). Increase in \( n \) yields a slight increase in \( \bar{Nu} \), showing better heat transfer performance for the dilatant fluid than for pseudoplastic fluid. This behavior can be explained by the fact that increasing \( n \) induces the same effect as reducing \( Da \), where the dimensionless bulk mean fluid temperature is decreased due to smaller dimensionless fluid temperature (smaller wall-fluid temperature difference).

![Figure 7](image)

**Figure 7.** Overall Nusselt number plotted as a function of Da for \( H = 20 \) and selected \( \gamma \) values for (a) \( n = 0.5 \) and (b) \( n = 2 \).
Conclusions
Forced convection heat transfer of non-Newtonian fluids in porous media has been investigated. The fluid flow is characterized by the Brinkman-extended Darcy equation modified for non-Newtonian fluids, and the two-equation model is employed for heat transfer between the fluid and solid phases. As there is no exact solution for arbitrary power-law index value obtainable for the momentum equation, the momentum integral boundary layer method has been used to attain the approximate velocity profile based on the parabolic model for the case of $\text{Da} \ll 1$, which is of practical engineering importance. The overall Nusselt number can be obtained from the analytical solutions of solid and fluid temperature distributions. The parabolic model is capable of providing temperature distributions that agree excellently with

![Figure 8. Overall Nusselt number plotted as a function of $H$ for $\text{Da} = 0.01$ and selected $\gamma$ values for (a) $n = 0.5$ and (b) $n = 2$.](image1)

![Figure 9. Overall Nusselt number plotted as a function of $n$ for $H = 20$, $\gamma = 0.05$, and $\text{Da} = 0.01$.](image2)

Conclusions
Forced convection heat transfer of non-Newtonian fluids in porous media has been investigated. The fluid flow is characterized by the Brinkman-extended Darcy equation modified for non-Newtonian fluids, and the two-equation model is employed for heat transfer between the fluid and solid phases. As there is no exact solution for arbitrary power-law index value obtainable for the momentum equation, the momentum integral boundary layer method has been used to attain the approximate velocity profile based on the parabolic model for the case of $\text{Da} \ll 1$, which is of practical engineering importance. The overall Nusselt number can be obtained from the analytical solutions of solid and fluid temperature distributions. The parabolic model is capable of providing temperature distributions that agree excellently with
the exact solution for the case of \( n = 1 \). The velocity distribution is found to be more uniform as either the Darcy number \( Da \) is reduced or the power-law index \( n \) value is increased. The temperature distributions are found to be sensitive to the non-dimensional interfacial heat transfer coefficient \( H \) and effective thermal conductivity ratio \( \gamma \), as well as \( Da \) and \( n \). The fluid temperature approaches the solid temperature when either \( \gamma \) or \( H \) is increased. In addition, decreasing \( Da \) and increasing \( n \) will produce a higher fluid temperature profile with a more uniform distribution where these subsequently yield a higher overall Nusselt number \( \overline{Nu} \). Two other two important parameters, \( H \) and \( n \), significantly affect \( \overline{Nu} \), where increasing \( H \) or decreasing \( \gamma \) will increase \( \overline{Nu} \) drastically. However, \( \overline{Nu} \) does not increase indefinitely but reaches an asymptotic value once \( H \) approaches infinity. Finally, the dependence of \( \overline{Nu} \) on \( n \) is found to reinstate the effect of using both the dilatant and pseudoplastic fluids. Despite simplification made in the momentum integral technique, the approximate solution based upon the parabolic model provides an avenue for practical application of the problem of non-Newtonian fluid flow in a porous medium with adequate accuracy.

**Nomenclature**

- \( a \): specific surface area, \( m^{-1} \)
- \( B_1 \) to \( B_6 \): coefficients in Equations (39) to (44)
- \( c_r \): heat capacity of fluid, \( J \cdot kg^{-1} \cdot K^{-1} \)
- \( C_1 \) to \( C_6 \): coefficients in Equations (27) to (32)
- \( d \): channel height, \( m \)
- \( D \): coefficient in Equation (33)
- \( Da \): Darcy number, \( (K^*/\epsilon^n)^{2/(1+n)}/d^2 \)
- \( E \): coefficient in Equation (35)
- \( h \): overall heat transfer coefficient, \( W \cdot m^{-2} \cdot K^{-1} \)
- \( h_{sf} \): interfacial heat transfer coefficient, \( W \cdot m^{-2} \cdot K^{-1} \)
- \( H \): nondimensional interfacial heat transfer coefficient
- \( k \): thermal conductivity, \( W \cdot m^{-1} \cdot K^{-1} \)
- \( K \): permeability for flow of Newtonian fluid, \( m^2 \)
- \( K^* \): intrinsic permeability of the porous medium for flow of power-law fluid
- \( n \): power-law index
- \( \overline{Nu} \): overall Nusselt number, \( \overline{\theta d}/k_{fe} \)
- \( p \): pressure, \( N \cdot m^{-2} \)
- \( P \): dimensionless pressure gradient, \( (K^*/\mu^*u_{m}^n\epsilon^n)(d\langle p \rangle_t/dx) \)
- \( q_w \): uniform wall heat flux, \( W \cdot m^{-2} \)
- \( T \): temperature, \( K \)
- \( u \): fluid velocity, \( m \cdot s^{-1} \)
- \( U \): dimensionless velocity, \( \langle u \rangle_t/u_m \)
- \( U_c \): dimensionless centerline velocity
- \( x \): axial coordinate, \( m \)
- \( y \): transverse coordinate, \( m \)
- \( Y \): dimensionless vertical coordinate, \( y/d \)
- \( \langle \rangle \): volume-averaged value over entire porous structure
- \( \langle \rangle_t \): volume-averaged value over fluid region
- \( \langle \rangle_s \): volume-averaged value over solid region
Greek Letters
\( \gamma \) effective thermal conductivity ratio, \( \varepsilon k_t/(1 - \varepsilon)k_s \)
\( \delta \) dimensionless boundary layer thickness
\( \varepsilon \) porosity
\( \theta \) dimensionless temperature
\( \lambda \) coefficient, in Equation (34)
\( \mu \) fluid viscosity, \( \text{N} \cdot \text{s} \cdot \text{m}^{-2} \)
\( \mu^* \) fluid consistency of the inelastic non-Newtonian power-law fluid, \( \text{N} \cdot \text{s}^n \cdot \text{m}^{-2} \)
\( \rho_f \) density of fluid

Subscripts
c centerline value
ex exact solution
f fluid
l linear model
m mean value
s solid
w wall

References


